Projected Written Notes from the M325K LECTURE ON THURSDAY, FEBRUARY 22, 2024, ON MATHEMATICAL INDUCTION and on STRONG MATHEMATICAL INDUCTION.

CLASS # 12

Mathematical Induction

Consider the formula $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for any integer $n \geq 1$.

This is an infinite List of statement..

P_n's

P_k's

$P_1: 1 = \frac{1(1+1)}{2} = \frac{(1)(2)}{2} = 1$

$P_2: 1+2 = \frac{2(2+1)}{2} = \frac{2\times 3}{2} = 3$

$P_3: 1+2+3 = \frac{3\times 4}{2} = 6$

The formula is proven true for $n = 1, 2, 3$

We use:

A proof by Mathematical Induction has 3 steps

Step ① Show that P_1 is true by hand. [The Basis Step]

Step ② Prove that, for every integer $k \geq 1$, if P_k is true, then P_{k+1} is true. [The Induction Step]

Step ③ Apply the principle of Mathematical Induction to prove P_n is true for all integers $n \geq 1$. [The final conclusion]

The Principle of Mathematical Induction

Let a be the first value of n to be considered and let P(n) be a predicate with n as the predicate variable.

IF

(1) P(a) is true (i.e, P(n) when n is set to its first value, a) and

(2) For every integer $k \geq a$,

If P(k) is true, Then P(k+1) is true,

THEN

P(n) is true for every integer $n \geq a$.

In a proof by Mathematical Induction, the whole proof involves proving that the "IF" conditions of the Principle of Mathematical Induction shown above are true.

The BASIS STEP is the part of the proof which proves condition (1).

The INDUCTIVE STEP is the part of the proof which proves condition (2).

After (1) and (2) have been proved, we can invoke the Principle of Mathematical Induction to conclude: "*$\therefore$ P(n) for every integer $n \geq a$ by the Principle of Mathematical Induction.*"

Mathematical Induction (in Diagram Form with a = 1):

Given an infinite list of statements:

$P_1, P_2, P_3, P_4, \ldots$

If we prove: (1) P_1, and

If we prove: (2) For all integers $k \geq 1$,
if P_k, then P_{k+1},

Then, we can conclude:
For all integers $n \geq 1$, P_n,
by the Principle of Mathematical Induction.

IF we prove :

that this is true, and that, for all integers $k \geq 1$,

If this is true, then this is true,

THEN . . .

$P_1, P_2, P_3, P_4, \ldots, P_k, \quad P_{k+1}, \quad \ldots$

We can logically conclude that ALL of these statements are true.

$P_1, P_2, P_3, P_4, \ldots, P_k, P_{k+1}, \ldots$

The Design for Proofs Using Mathematical Induction (1st Principle)
(*a* represents a particular integer.)

To Prove: For every integer n such that $n \geq a$, predicate P(n).

Proof: (by Mathematical Induction)

[The Basis Step shows that P(n) is true when n is replaced by *a*, the first value of n. Often, $a = 1$ or $a = 0$.]

Let n = *a* .

. . . (calculations with n = *a*) . . . ;

∴ For n = *a*, P(n) . [End of Basis Step]

[The Inductive Step proves that, for every integer k such that $k \geq a$, **if** P(k) , **then** P(k+1)" .]

Let k be any integer such that $k \geq a$.

Suppose P(k). [Inductive Hypothesis]

[N.T.S.: P(k+1).] [That is, we N.T.S. that P(n) is true when n is replaced by k+1.]

....

. . . (proof statements, one of which is justified by the Inductive Hypothesis supposition)

....

∴ P(k+1).

∴ For every integer k such that $k \geq a$, **if** P(k) , **then** P(k+1) by Direct Proof.
[End of Inductive Step]

∴ For every integer n such that $n \geq a$, P(n), by the Principle of Mathematical Induction.
QED

The last statement shown here in the Inductive Step,

"For every integer k such that $k \geq a$, **if** P(k) , **then** P(k+1), by Direct Proof, "

is **required** by Dr. Shirley to be written as the last statement of any Inductive Step (because it says that that the Inductive Step has validly served its purpose and has done so using the method of Direct Proof). Dr. Shirley refers to it as **"The Required Last Statement of the Inductive Step"** when he is grading these proofs.

Also, the statement "**Suppose P(k)**" in the Inductive step is called ***the Inductive Hypothesis***. The one and only required comment in this class is one which identifies this statement as being the Inductive Hypothesis of the proof: **"Suppose P(k). [Inductive Hypothesis]"**

Whenever a conclusion is justified by this supposition, this is indicated by saying:

" ∴ Such-and-such, ***by the Inductive Hypothesis***. "

Note: The first statement of the Inductive Step, shown above, has the wording
"Let k be any integer such that $k \geq a$,"
Using **this** wording of the sentence to begin the Inductive Step is also **required**.

The author of the textbook uses a different wording of the sentence to begin the Inductive Step. The wording that the author uses most often to begin an Inductive Step is:

"Suppose that P(k) is true for some integer $k \geq a$.
[We must show that P(k+1) is true.] " **NOT ALLOWED**

You may not use this wording to begin the Inductive Step. You must use the required wording **"Let k be any integer such that $k \geq a$"** to begin the Inductive Step.

A First Example of a Proof using the method of Mathematical Induction

Theorem 5.2.2: For every integer $n \geq 1$, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Proof: (by Mathematical Induction)

[Basis Step] Let n = 1. $\therefore 1 + 2 + 3 + \ldots + n = 1$, and

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1.$$

$\therefore$ For n = 1, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ by substitution. [End of the Basis Step]

The Inductive Step here proves the statement "For every integer $k \geq 1$,
if $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$,
then $1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$. [<-- Big Comment]

[Inductive Step]

Let k be any integer such that $k \geq 1$.

Suppose that $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$. [Inductive Hypothesis]

We need to show that $1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$.

By the Inductive Hypothesis, we can substitute $\frac{k(k+1)}{2}$ for $1 + 2 + 3 + \ldots + k$. [<--Big Comment]

SEE THE COMPLETION OF THE PROOF ON THE NEXT PAGE.

The Completion of The Proof:

[NTS: $1+2+\cdots+k+k+1 = \frac{(k+1)((k+1)+1)}{2}$.]

By the Inductive Hypothesis, and ROA,

$$(1+2+3+\cdots+k)+k+1 = \frac{k(k+1)}{2}+k+1.$$

$$= \frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}.$$

$\therefore 1+2+3+\cdots+k+k+1 = \frac{(k+1)((k+1)+1)}{2}$ by Transitivity

$\therefore$ For all integers $k \geq 1$, if $1+2+\cdots+k = \frac{k(k+1)}{2}$,

then $1+2+\cdots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$,

by Direct Proof.

[END of Inductive Step]

$\therefore$ By Mathematical Induction, for all integers $n \geq 1$, $1+2+\cdots+n = \frac{n(n+1)}{2}$.

QED.

A Math-Speak Shorthand

Consider this statement!

" $3^2 < 100$

and $4^2 < 100$

and $5^2 < 100$

and $6^2 < 100$".

EQUIVALENT SHORTHAND!

" $t^2 < 100$

for all integers t

such that

$3 \le t \le 6$."

The Principle of Strong Mathematical Induction (2^{nd} Principle)

Let a and b be fixed positive integers such that $a \leq b$, and let $P(n)$ be a predicate for all integers $n \geq a$.

IF

1) $P(a), P(a+1), \ldots, P(b)$ are all true, and
2) For every integer $k \geq b$,

   If $P(m)$ is true for every integer m such that $a \leq m \leq k$, then $P(k+1)$ is true,

THEN

$P(n)$ is true for every integer n such that $n \geq a$.

Strong Mathematical Induction (in Diagram Form) with $a = 1$:

Given an infinite list of statements: $P_1, P_2, P_3, P_4, \ldots$

IF we prove that:

<u>These</u> are true, and that, for all integers $k \geq b$, (so $k+1 \geq b+1$)

if all the P_m, with $1 \leq m \leq k$, are true, | then P_{k+1} is true,

. . . .

$P_1, P_2, \ldots, P_{\mathbf{b}}, P_{\mathbf{b+1}}, \ldots, P_m, \ldots, P_{k-1}, P_k, P_{k+1}, \ldots$

THEN we can logically conclude that <u>ALL of these</u> statements are true.

. . . . . .

$P_1, P_2, P_3, P_4, \ldots, P_{k-1}, P_k, P_{k+1}, \ldots$

The Design for Proofs using the Principle of Strong Mathematical Induction (2^{nd} Principle)

(a represents a particular integer.)

To Prove: For every integer n such that $n \geq a$, predicate P(n) .

Proof: (by Strong Mathematical Induction)

[**Basis Step: Show that P(n) is true when n = *a*, *a* + 1, *a* + 2, ..., *b* for an appropriate number b $\geq$ *a*.**]

. . . [See below for a format for the Basis Step.] **[End of Inductive Step]**

[Inductive Step:

To Prove: For every integer k $\geq$ b,

if P(m) is true for every integer m such that a $\leq$ m $\leq$ k , then P(k+1) is true.]

Let k be any integer such that k $\geq$ b .

Suppose P(m) , for every integer m such that $a \leq m \leq k$. [This is the ***Inductive Hypothesis***.]

[NTS: P(k+1)]

. . . (proof statements come here,
including statements that apply the Inductive Hypothesis for various integers t as needed,
that is, statements similar to the following :

"Since $\boldsymbol{a} \leq \mathbf{t} \leq \mathbf{k}$, P(**t**) , by the Inductive Hypothesis."

[Note: $\boldsymbol{a} \leq \mathbf{t} \leq \mathbf{k}$ needs to be proved first!])

. . .

$\therefore$ P(k+1) .

$\therefore$ For every integer k such that k $\geq$ b , if P(m) , for every integer m
such that $a \leq m \leq k$, then P(k+1), by Direct Proof. **[End of Inductive Step]**

$\therefore$ P(n) for every integer n such that n $\geq$ a, **by Strong Mathematical Induction**. **QED**

A Format for the Basis Step

Let n = a;	. . . (calculations with n = a) . . . ;	$\therefore$ For n = a,	P(n) .
Let n = a + 1;	. . . (calculations with n = a + 1) . . . ;	$\therefore$ For n = a + 1,	P(n) .
Let n = a + 2;	. . . (calculations with n = a + 2) . . . ;	$\therefore$ For n = a + 2,	P(n) .
. . .	. . .	. . .	
Let n = b;	. . . (calculations with n = b) . . . ;	$\therefore$ For n = b,	P(b) . [End of Basis Step]

Notes:

1) The number of initial cases of the predicate that need to be verified in the Basis Step depends on the nature of the predicate P(n) . Sometimes there will be only one initial case to verify. Usually, however, there are two or more cases. The number of cases that need to be verified in the Basis Step often equals the number of previous cases (P(k), P(k-1), P(k-2), etc.) that need to be accessed in order to prove P(k+1) .

2) The supposition " Suppose that P(m) is true for every integer m such that $a \leq m \leq k$ " is how mathematicians say

"Suppose that P(k) and all of the cases of the predicate before P(k) are true."

This supposition is called *the Inductive Hypothesis*. When justifying a conclusion by this supposition, you indicate this by saying: " $\therefore$ Such-and-such is true, ***by the Inductive Hypothesis***. "

A Proof by Strong Mathematical Induction

The sequence (a_n) is defined as follows:

$a_0 = 1$, $a_1 = 3$, $a_2 = 5$, and,

for all integers $t \geq 3$, $a_t = 3a_{t-2} + 2a_{t-3}$.

To Prove: For all integers $n \geq 0$, $a_n < 2^{n+1}$.

Proof: [By STRONG MATH'L INDUCTION]

[BASIS STEP] Let $n=0$. $a_n = a_0 = 1$ and $2^{n+1} = 2^{0+1} = 2$

Since $1 < 2$, $a_n < 2^{n+1}$ for $n=0$.

Let $n = 1$. $\therefore a_n = a_1 = 3$ and $2^{n+1} = 2^{1+1} = 4$.

Since $3 < 4$, $a_n < 2^{n+1}$ for $n = 1$.

Let $n = 2$. $\therefore a_n = a_2 = 5$ and $2^{n+1} = 2^{2+1} = 2^3 = 8$.

Since $5 < 8$, $a_n < 2^{n+1}$ for $n = 2$.

[END OF BASIS STEP]

[INDUCTIVE STEP]

Let k be any integer such that $k \geq 2$.

Suppose $a_m < 2^{m+1}$, for all integers m such that $0 \leq m \leq k$. [The Inductive Hypothesis]

[NTS: $a_{k+1} < 2^{(k+2)}$].

(Proof continued on the next page.)

The Proof Continues!

[Recall that we NTS $a_{k+1} < 2^{(k+2)}$.].

[Note: we want to say that $a_{k+1} = 3a_{k-1} + 2a_{k-2}$,

but we can't say this until we prove that the subscript $k+1 \geq 3$.]

Since $k \geq 2$, $k+1 \geq 3$.

$\therefore$ By the formula in the sequence definition, since $k+1 \geq 3$, $a_{k+1} = 3a_{k-1} + 2a_{k-2}$.

[Next, we want to apply the inductive hypothesis to a_{k-1} and to a_{k-2}, but to do that, we must prove that $0 \leq k-2 \leq k$ and $0 \leq k-1 \leq k$.]

Since $k \geq 2$, $k-2 \geq 0$. $\therefore 0 \leq k-2 < k-1 \leq k$.

Since $0 \leq k-2 \leq k$, $a_{k-2} < 2^{k-1}$ by the Inductive Hyp.

Since $0 \leq k-1 \leq k$, $a_{k-1} < 2^{k}$ by the Inductive Hypothesis.

Recall that $a_{k+1} = 3a_{k-1} + 2a_{k-2}$.

$\therefore$ $3a_{k-1} < 3 \times 2^{k}$ and $2 \cdot a_{k-2} < 2 \cdot 2^{k-1} = 2^{k}$.

So, $3a_{k-1} + 2a_{k-2} < 3 \times 2^{k} + 2 \times 2^{k-1} = 3 \times 2^{k} + 2^{k}$

So, $3a_{k-1} + 2a_{k-2} < 3 \times 2^{k} + 2^{k} = 4 \times 2^{k} = 2^{k+2}$ since $4 = 2^{2}$.

$\therefore$ $a_{k+1} < 2^{k+2}$, by substitution.

$\therefore$ For all integers $k \geq 2$, if $a_m < 2^{m+1}$ for all integers such that $0 \leq m \leq k$, then $a_{k+1} < 2^{k+2}$, by direct proof.

[END OF INDUCTIVE STEP]

[The FINAL CONCLUSION, APPLYING the Principle of STRONG INDUCTION]:

Therefore, for all integers $n \geq 0$, $a_n < 2^{n+1}$, by the Principle of Strong Mathematical Induction.

Q.E.D